FNC010,FNC013 - problem sheet 1 solutions

August 6, 2005

Limits

1.

$$\begin{aligned} \text{if: } |f(x) - 7| = |2x - 2| &< \epsilon \\ \therefore |x - 1| &< \frac{\epsilon}{2} \\ \therefore \delta &= \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned} \text{if: } |(x + 3)(x - 1) - 5| &< \epsilon \\ \therefore |x^2 + 2x - 8| = |(x - 2)||(x + 4)| &< \epsilon \\ \therefore |x - 2| &< \frac{\epsilon}{|x + 4|} \\ \therefore \delta &= \frac{\epsilon}{|x + 4|} \end{aligned}$$

2.

$$\begin{aligned} \sin\left(\frac{1}{x} - \frac{1}{2}\right) &= \sin\frac{1}{x}\cos\frac{1}{2} + \sin\frac{1}{2}\cos\frac{1}{x} \\ \therefore \lim_{x\to 2}\sin\left(\frac{1}{x} - \frac{1}{2}\right) &= \sin\frac{1}{2}\cos\frac{1}{2} + \sin\frac{1}{2}\cos\frac{1}{2} = 0 \end{aligned}$$

$$\begin{aligned} \frac{x^2 - 9}{x + 3} &= \frac{(x + 3)(x - 3)}{x + 3} = (x - 3) \\ \therefore \lim_{x\to 5}\frac{x^2 - 9}{x + 3} &= 2 \end{aligned}$$

Derivatives

1.

$$\begin{aligned}\frac{d(x^3)}{dx} &= \lim_{h\to 0}(x+h)^3 - x^3 h \\ &= \frac{3hx^2+3h^2x+h^3}{h} \\ &= 3x^2\end{aligned}$$

$$\begin{aligned}\frac{d(\cos x)}{dx} &= \lim_{h\to 0}\frac{\cos x + h - \cos x}{h} \\ &= \frac{\cos x\cos h - \sin h \sin x - \cos x}{h}\end{aligned}$$

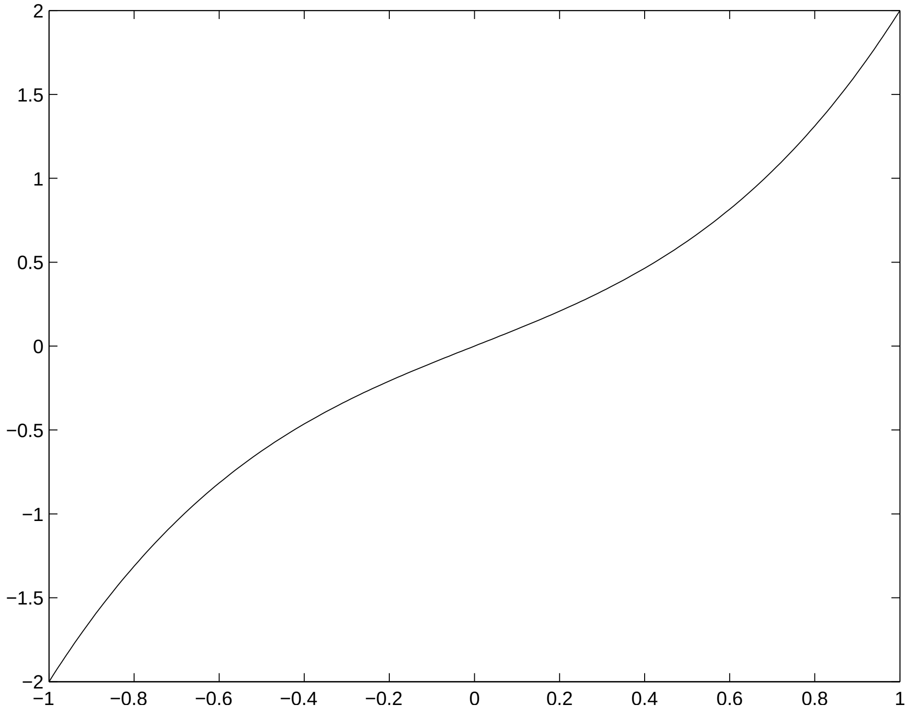
At very small values of h, $\cos h$ approaches 1, and $\sin h$ approaches h, therefore:

$$\begin{aligned}\therefore \frac{\cos x\cos h - \sin h\sin x - \cos x}{h} &= \frac{\cos x - \sin h \sin x - \cos x}{h} \\ &= -\frac{h\sin x}{h} = -\sin x\end{aligned}$$

2. (a)

$$\begin{aligned}\frac{d(x^2+\cos x^2)}{dx} &= 2x - 2x\sin x^2 \\ &= 2x(1-\sin x^2) \\ \frac{d\frac{x^2+1}{x}}{dx} &= \frac{x.(2x)-(x^2+1)}{x^2} \\ &= \frac{x^2-1}{x^2} \\ \frac{d(\sin x^2\cos 2x)}{dx} &= 2x\cos x^2 + \sin x^2 2\sin(2x) \\ \frac{dy}{dx} &= \frac{dy}{du}.\frac{du}{dx} = -\sin(\sin x).\cos x\end{aligned}$$

(b) $f'(x) = 3x^2 + 1$



The form for this curve could have been inferred in a number of ways. For example, the shape of $f(x) = x^3$ will be similar except that the gradient would have been zero at $x = 0$. In this case because of the added x term, the inflection point has a none zero gradient. The minimum value of the gradient can be inferred from the expression for $f'(x)$ - since the minimum of x^2 is zero, the lowest that $f'(x)$ can go is thus 1.

3. To make the function continuous, concentrate at its values around $x = 1$, making sure that the two constituent functions meet up:

$$
\begin{aligned}
(1)^2 + 5 &= A(1) - 5 \\
\therefore A &= 6 + 5 = 11
\end{aligned}
$$